Credit card APRs and monthly interest rates

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Abstract

The APR, or annual percentage rate, is a figure that is supposed to represent the total cost for credit for any particular type of borrowing. This paper demonstrates that, for fee free revolving credit products, the APR is largely a mathematical construct, but one in which the APR and monthly interest rate are different representations of the same number. Neither monthly rate nor APR convey much information about the actual cost for any particular pattern of credit card use.
1 Introduction

This paper shows the equivalence of the monthly interest rate and the APR on a credit card. The only proviso is that the card does not carry a fee (the relationship does not hold in the case of products that have are fee bearing). It also describes the assumptions needed to calculate credit card APRs, some limitations of the APR itself, and some details of the APR calculation as it relates to credit cards.

There are two strands to the argument. The first is to show that the APR can be calculated from a monthly rate. The second is to show that this monthly rate is the same as the actual monthly interest rate charged on credit cards. In order for the ‘calculated’ and ‘actual’ monthly rates to be identical, the same – unrealistic – assumptions must be imposed on card holders’ behaviour.

The Consumer Credit Directive (The European Union, 2008) states that the ‘annual percentage rate of charge’ means the total cost of the credit to the consumer, expressed as an annual percentage of the total amount of credit’. This might seem like an interest rate, but it is not, for a variety of reasons. Firstly, it may include charges other than interest rate and secondly, it depends on the balance over the term of the loan. A good illustration of this is given by The UK Cards Association (2010b), which shows two £1,000 loans repaid over a year, both of which require total repayments of £1,200. The different payment schedules result in APRs of 20% and 41.3%, despite the fact that the loans each cost the same cash amount and were repaid in full in one year.

2 Summary and conclusions

For fee free cards, the APR and the monthly interest rate charged on the card are simply re-expressions of one another. In other words, if we know one, we know the other.
The relationship between the two numbers – monthly interest rate and APR – is purely mathematical, and is a consequence of the calculation. Neither number will necessarily bear any relationship to the amount any particular card holder will pay on their card.

3 The APR equation

The APR equation is laid down in law by The European Union (The European Union, 2008), and is repeated in the UK Statutory Instrument (SI 2010 No. 1011, 2010), as follows.

\[ \sum_{k=1}^{m} C_k(1 + X)^{-t_k} = \sum_{l=1}^{m'} D_l(1 + X)^{-s_l} \]  

(1)

The Office of Fair Trading also produced equation 2 (Office of Fair Trading, 2007). This publication does not seem to be on the OFT’s website any longer, but it is referred to elsewhere on the site.

\[ \sum_{k=m}^{K} A_k/(1 + i)^{tk} = \sum_{K'=m'}^{k} A'_{K'}/(1 + i)^{t_{K'}} \]  

(2)

The important point about these equations is that they express an equivalence between the sum of present values of future drawdowns on the left hand side and the sum of present values of future repayments on the right hand side. Generally speaking, the calculation is implemented by assuming that there is only one drawdown of credit, so there is only one term on the left hand side. Indeed, the European Union (The European Union, 2008) noted that equation 1 can be re-expressed as in equation 3.

\[ S = \sum_{k=1}^{n} A_k(1 + X)^{-t_k} \]  

(3)

In theory, there is no reason why a ‘composite APR’ could not be created for any particular individual. However, interpreting the APR that resulted would be problematical, because it would be a combination of all secured, unsecured and revolving credit taken by a person. In the rest of this document, it is assumed that such a ‘composite APR’ is, in reality, largely meaningless, so only one figure needs to appear on the left hand side of equation 1 as shown in equation 3. S, therefore, is simply the loan amount, and the value of X that results in the right hand side of equation 3 being equal to S, will then be the required APR.

4 Credit card assumptions

Credit cards are, by their nature, revolving credit products. This means that, when calculating credit card APRs according to equation 3, additional assumptions about usage need to be made, as well as the 13 other assumptions listed in SI 2010 No. 1011 (2010) about timing, variable interest rates and so on. The relevant ‘extra credit card assumptions’ are as follows.

1. A ‘loan’ of £1,200 is taken if the credit limit is not known
2. The ‘loan’ is provided for a period of one year
3. It will be repaid in 12 equal monthly instalments
4. Each month is assumed to be \( \frac{1}{12} \) of a year (i.e. 30.4166\ldots days)

5 Some limitations of the APR

The APR calculation has limitations that are not just restricted to credit cards. It may not be straightforward to interpret APRs for fixed repayment schedule loans either. In particular, APRs for two products can only be used to compare costs if, and only if, the amounts borrowed and the repayment schedules are identical. Even then, the APR cannot easily be used to assess the cost, in pounds sterling, of any particular product.

Credit cards have an additional set of problems, arising from the flexible nature of the product and the wide variety of ways in which they can be used.

Anyone who has studied individual consumer behaviour knows that it can be highly variable across any group of people – for some examples, see Hand and Blunt (2001) or Stewart (2009). These examples of different behaviour patterns put into context the simplicity of the assumptions described in section 4. Hardly anyone – probably no-one – will actually use their card in the way that is assumed in the APR calculation.

6 The calculation

It should be clear that the APR for any given credit card can be calculated as shown in equation 4. This is simply equation 3 where there are 12 repayments. Note that \( A \) can be calculated from the interest rate (usually applied monthly) that is charged on the card, and also that \( A \), not \( A_k \), appears in the numerator of each term on the right hand side, because of the ‘12 equal monthly instalments’ assumption.

\[
S = \sum_{k=1}^{12} A(1 + X)^{-t_k} \tag{4}
\]

Expanding equation 4 gives equation 5, which, when multiplied by \((1 + X)^{-\frac{1}{12}}\) results in equation 6.

\[
S = A(1 + X)^{-\frac{1}{12}} + A(1 + X)^{-\frac{2}{12}} + \cdots + A(1 + X)^{-\frac{12}{12}} \tag{5}
\]

\[
(1 + X)^{-\frac{1}{12}} S = A(1 + X)^{-\frac{2}{12}} + A(1 + X)^{-\frac{3}{12}} + \cdots + A(1 + X)^{-\frac{12}{12}} \tag{6}
\]

Subtracting equation 6 from equation 5 gives equation 7.

\[
(1 - (1 + X)^{-\frac{1}{12}})S = A \left( (1 + X)^{-\frac{1}{12}} + (1 + X)^{-\frac{2}{12}} + \cdots + (1 + X)^{-\frac{12}{12}} \right) -
A \left( (1 + X)^{-\frac{2}{12}} + (1 + X)^{-\frac{3}{12}} + \cdots + (1 + X)^{-\frac{12}{12}} \right) \tag{7}
\]
Most of the terms on the right hand side of equation 7 cancel each other, to give equation 8.

\[
S = \frac{A(1 + X)^{-\frac{1}{12}} - A(1 + X)^{-\frac{13}{12}}}{1 - (1 + X)^{-\frac{1}{12}}} = \frac{A(1 + X)^{-\frac{1}{12}} (1 - (1 + X)^{-\frac{13}{12}})}{1 - (1 + X)^{-\frac{1}{12}}}
\]  

(8)

Note that equation 8 has been shown in this way, in 12th roots of \(X\), because of the relationship between this and the monthly rate. This will be relevant on returning to this, in equation 10 below, which will show a similar equation.

If we now let \(x\) be the monthly rate, the APR equation (4) can be expressed as equation 9, where \(x\) and \(k\) have replaced \(X\) and \(t_k\).

\[
S = \sum_{k=1}^{12} A(1 + x)^{-k}
\]  

(9)

Performing algebraic manipulations similar to the ones I showed in equations 5 to 8, but starting with equation 9 instead, we have equation 10.

\[
S = \frac{A(1 + x)^{-1} \left( 1 - (1 + x)^{-12} \right)}{1 - (1 + x)^{-1}}
\]  

(10)

Note the similarity of equations 4 and 9, which are, mathematically equivalent. We can either solve for a monthly rate \(x\) and use months periods, or an annual rate \(X\) (or APR), and use time periods expressed as fractions of a year (i.e. \(\frac{1}{12}\)s of a year). Similarly, equations 8 and 10, which were derived from equations 4 and 9, are equivalent. They are related by the simple equation shown in 11, remembering that \(X\) is the APR and \(x\) is a monthly rate.

\[
(1 + X) = (1 + x)^{12}
\]  

(11)

In other words, if we know the monthly rate, we can calculate – and simply – the APR. And vice versa too.

The monthly rate just described might be equally as flawed as the APR (i.e. it could be a simple consequence of the nature of the calculation). We need to show what happens if we start with the actual interest rate that is applied to the credit card, and what relationship this might have, or not, to the calculated APR.

7 Credit card monthly rates

Interest is typically charged using a monthly rate, which will be given in each card’s terms and conditions. This monthly rate is applied to the average daily balance in the month.\(^1\) The point about ‘average daily balance’ is important, because this is a figure that is calculated by

\(^1\)Note that a daily rate may be applied to daily balances, but the two methods are equivalent, and can be converted easily from one to the other.
averaging the balance across each day of the month, for the number of days in the month. For example, if a card holder had a balance of £1,500 for 15 days then made a repayment of £1,000 and had a balance of £500, the average daily balance (in this thirty day month) would be as shown in equation 12.

\[
\frac{(1,500 \times 15) + (500 \times 15)}{30} = 1,000 \tag{12}
\]

There may be slight variations to this simple equation, depending on the length of the month, where weekends and bank holidays appear, and so on; these are unimportant for our purposes here, because the APR calculation assumes ‘standard length time periods’ as described in [European Union (2008) and SI 2010 No. 1011 (2010)]. And, more importantly, the effects will even out over a year for any particular card holder.

In any given month \( t \), assuming that every transaction happens at monthly intervals – to be consistent with the APR assumptions – a balance of \( s_t \), a repayment of \( a \) and a monthly interest rate of \( r \) are related by equation 13.

\[
s_t = s_{t-1} - a + rs_{t-1} \tag{13}
\]

In other words, the balance in one month is completely determined by the balance in the previous month, the repayment, and the interest rate. \(^2\) For the APR, we are interested in a set of ‘transactions’ where £1,200 is taken on day 1 and repaid in 12 equal monthly instalments.

We now have a situation in which we have \( s \) and we know \( r \) from the terms and conditions on the card. Further, we have a recursive relationship where the balance in one month depends on the repayment and the balance at the end of the month; we can solve this for any particular number months (12 in the case of replicating the APR), as follows, with the following two months shown in equations 14 and 15. Here, \( s_1 \) replaces \( s_t \) for the first month’s end balance. In this case, \( s_0 \) is the balance at the very beginning of the agreement, which will be the assumed credit limit.

\[
s_1 = s_0 - a + rs_0
\]

\[
= s_0(1 + r) - a \tag{14}
\]

\[
s_2 = s_1(1 + r) - a \tag{15}
\]

However, we can substitute for \( s_1 \) from equation 14 into 15, resulting in equation 16.

\[
s_2 = (s_0(1 + r) - a)(1 - r) - a
\]

\[
= s_0(1 + r)^2 - a(2 + r) \tag{16}
\]

Similar manipulation for month three results in equation 17.

\(^2\)This ignores, of course, that most card holders make several transactions every month, but we need to assume this to try to replicate the conditions imposed by the APR assumptions. Given that there were, in 2010, more than 2 billion transactions across more than 50 million accounts, this is clearly unrealistic (‘Credit Card Monthly Release Tables’ at [http://www.bba.org.uk/statistics](http://www.bba.org.uk/statistics)). And, of course, transactions may be for any amount, and may occur at any time in the year – both of which will influence cost.
\[ s_3 = s_0(1 + r)^3 - a(3 + 3r + r^2) \]  

A pattern is starting to emerge for the second term on the right hand side of these equations, which is shown in equation [18] so we use this to state the equation for the balance at the end of month 12 in equation [19].

\[ a \frac{(1 + r)^n - 1}{r} \]  

\[ s_{12} = s_0(1 + r)^{12} - a \left( \frac{(1 + r)^{12} - 1}{r} \right) \]  

However, given the assumptions required by the APR equation, as shown in section 4, we know that the balance at the end of month 12 must be zero, so equation [19] becomes equation [20].

\[ a = \frac{s_0(1 + r)^{12}}{(1 + r)^{12} - 1} \]  

\[ a = \frac{r s_0(1 + r)^{12}}{(1 + r)^{12} - 1} \]  

Comparing equation [20] with equation [8] we see that \( a \equiv A \) and \( s_0 \equiv S \), so we can substitute from the former into the latter in equation [21].

\[ S = \frac{rS(1 + r^{12})}{1 - \left( 1 + X^{-1} \right)} \]  

Notice that \( S \) appears on both sides of this equation, so falls out of the calculation, and the equation can be rearranged to that \( r \) and \( X \) are on the left and right hand sides of equation [22] respectively.

\[ \frac{(1 + r)^{12} - 1}{r (1 + r)^{12}} = \frac{1 - (1 + X)^{-1}}{(1 + X)^{12} - 1} = \frac{(1 + X) - 1}{(1 + X)^{12} - 1} \]  

Now we let \( \alpha = (1 + r)^{12} \), and because \( r = \alpha^{\frac{1}{12}} - 1 \), we can re-write equation [22] as equation [23].

\[ \frac{\alpha - 1}{(\alpha^{\frac{1}{12}} - 1) \alpha} = \frac{(1 + X) - 1}{((1 + X)^{\frac{1}{12}} - 1)(1 + X)} \]  

It obvious from this that \( \alpha = (1 + X)^{\frac{1}{12}} \), which leads to the concluding equation [24].

\[ (1 + r)^{12} = (1 + X) \]  

In other words, assuming the (unrealistic) behaviour described in SI 2010 No. 1011 (2010), the calculated APR is the monthly rate charge to the power of 12.
8 Rounding

Note the implications of rounding have not been discussed. The actions to be taken are described in [SI 2010 No. 1011 (2010)], but they are ignored during the calculation of an APR until the very end of the calculation, when the APR is shown to 1 decimal place (and as a percentage).

If rounding is imposed at intermediate stages of the calculation, there may be small impacts on the APR, but typically they will not be seen until the second or third decimal place. For example, with a loan of £1,200, 12 equal monthly repayments and a monthly rate of 1.167%, the monthly repayment (calculated to 12 decimal places!) is £107.746799636238. . . . If this were to be rounded to £107.75 in 11 months, the final payment would need to be reduced to £107.70. In some cases, with different interest rates or loan amounts, such changes might affect the APR by 0.1% or so, but it would need to be an unusual product for the impact to be more than this in most cases.

9 Further work

In all of the preceding work, 12 equal monthly repayments were assumed, because of the assumptions necessary to carry out the APR calculation. A further enhancement would be to see if the equivalence of monthly interest rate and APR extended to different repayment schedules. This could include, for example, repayment over a fixed number of years, or with varying payments in each month.

Note that if any number of equal monthly repayments \( n \), rather than 12, were assumed, the above calculations would still hold.

However, if the terms on the right hand side of equation \([I]\) were of the form \( A_t(1 + X)^{-t} \), where \( A_t \neq A_{t+1} \neq A_{t+n} \) (remember that \( A_t \) is the repayment amount \( A \) in month \( t \)), then the mathematics of the subsequent series would become somewhat more complex.

The problem would be that of choosing a ‘representative’ set of assumptions about repayment behaviour. For example, how much will be repaid each month? When in the month should the repayment be made? And so on. The UK Cards Association’s response to A Better Deal for Consumers ([The UK Cards Association](2010)] showed that there is a wide range of use and repayment behaviour patterns among credit card holders.

Any simplistic assumptions would be likely to be as flawed as the ‘repayment in 12 equal monthly payments’ assumption currently used. Therefore, at the time of writing, it is unclear whether extending this work to more than ‘12 equal monthly repayments’ has any value.

References


SI 2010 No. 1011 (2010). The Consumer Credit (Total Charge for Credit) Regulations.

